1) $f(x,y)=4x+2y$

First rewrite to see how to use the power rule.

$$f(x,y)=4x^{1}+2y$$

Now differentiate term by term.

$$f_{x}(x,y)=\frac{\partial}{\partial x}(4x+2y)=\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial x}(2y)=4(1)x^{1-1}=4(1)x^{0}=4(1)=4$$

Notes:

1) $x^{0}=1$
2) With respect to x, $2y$ is treated as constant, so it differentiaties aways to 0.
3) I've shown you a very detailed approach. However, it amounts to writing "4" as the partial with respect to x.

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2) $f(x,y)=4x+2y$

First rewrite to see how to use the power rule.

$$f(x,y)=4x+2y^{1}$$

Now differentiate term by term, applying the power rule to the term with y.

$$f_y\ (x,y)=\frac{\partial}{\partial y}\left(4x+2y\right)=\frac{\partial}{\partial y}(4x)+\frac{\partial}{\partial y}(2y)=2(1)y^{1-1}=2y^{0}=2(1)=2$$

Notes:

1) $y^{0}=1$
2) With respect to y, 4x is treated as constant, so it differentiaties aways to 0.
3) I've shown you a very detailed approch, but it amounts to just writing "2" as the partial with respect to y.

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3) $f(x,y)=4x^2+2y$

Apply the power rule, as shown below.

$$f_x\,(x,y)=\frac{\partial}{\partial x}(4x^2+2y)=\frac{\partial}{\partial x}(4x^2)+\frac{\partial}{\partial x}(2y)=4(2)x^{2-1}=8x^1=8x$$

Notes:

1) With respect to x, $2y$ is treated as constant, so it differentiaties aways to 0.
2) I've shown you a very detailed approch, but it amounts to just writing "8x" as the partial with respect to x.

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4) $f(x,y)=4x^2+2y^3+4$

Apply the power rule, as shown below.

$$f_y\ (x,y)=2(3)y^{3-1}=6y^2$$

Notes:

1) With respect to y, $4x^2$ and 4 are treated as constant, so they differentiate away to 0.
2) I've shown you a very detailed approch, but it amounts to just writing "$6y^2$" as the partial with respect to y.

5) $f(x,y)=xy$

Rewrite first to see how to apply the power rule.

$$f(x,y)=x^{1}y$$

Apply the power rule, as shown below.

$$f_x\,(x,y)=\frac{\partial}{\partial x}(xy)=y\frac{\partial}{\partial x}(x)=y\cdot 1x^{1-1}=y\cdot 1\cdot x^{0}=y\cdot 1=y$$

Notes:

1) With respect to x, y is treated like a constant, so it stays unchanged. Pretend it's 5, for example. Factor it out.

2) $x^{0}=1$

3) I've shown you a very detailed approch, but it amounts to just writing "y" as the partial with respect to x.

6) $f(x,y)=x^3y-x+4$

Rewrite first to see how to apply the power rule.

$f(x,y)=x^3y-x^1+4$

Apply the power rule, as shown below.

$f_x\ (x,y)=3x^{3\text{-}1}y-1x^{1\text{-}1}=3x^2-x^0=3x^2-1$

Notes:

1) With respect to x, y is treated like a constant, so it stays unchanged. Pretend it's 5, for example.

2) $x^0=1$

3) The 4 disappears because with respect to x, its rate of change is 0.

7) $f(x,y)=x(x+y^2)$

You can begin by distributing the x into the parenthesis.

$$f(x,y)=x\cdot x+x\cdot y^2=x^2+xy^2$$

Apply the power rule, as shown below.

$$f_y(x,y)=2xy^{2-1}=2xy^1=2xy$$

Notes:

1) With respect to y, x is treated like a constant, so it stays unchanged. Pretend it's 5, for example.
2) Because x^2 doesn't multiply y, it differntiates away to 0.

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8) $f(x,y)=\frac{x}{y}$

You can begin by rewriting, as shown below.

$$f(x,y)=\frac{x}{y^{1}}=x\cdot y^{-1}=xy^{-1}$$

Apply the power rule, as shown below. Do not change x.

$$f_y\ (x,y)=\frac{\partial}{\partial y}(xy^{-1})=x\frac{\partial}{\partial y}(y^{-1})=(x)(-1)(y^{-1-1})=-xy^{-2}=\frac{-x}{y^{2}}$$

Notes:

1) With respect to y, x is treated like a constant, so it stays unchanged. Pretend it's 5, for example.

9) $f(x,y)=\ln(xy)$

This one is done by using the chain rule. First, you put xy in the denominator, and then multiply by the derivative of xy with respect to x.

$$f_x(x,y)=\frac{1}{xy}\left(\frac{\partial}{\partial x}(xy)\right)=\frac{1}{xy}(y)=\frac{1}{x}$$ The y's cancel.

10) $f(x,y)=x\sqrt{y}$

You can begin by rewriting, as shown below.

$$f(x,y)=x\cdot y^{\frac{1}{2}}$$

Apply the power rule, as shown below. Do not change x.If

you attempt to change x, God will be angry!

$$f_y\ (x,y)=\left(\frac{1}{2}\right)(x)\left(y^{\frac{1}{2}-1}\right)=\left(\frac{1}{2}\right)(x)\left(y^{\frac{1}{2}-\frac{2}{2}}\right)=\left(\frac{1}{2}\right)(x)\left(y^{-\frac{1}{2}}\right)=\frac{1}{2}x\cdot\frac{1}{y^{\frac{1}{2}}}=\frac{1x}{2\sqrt{y}}=\frac{x}{2\sqrt{y}}$$

Notes:

1) With respect to y, x is treated like a constant, so it stays unchanged. Pretend it's 5, for example.

11) $f(x,y)=ye^{x}$

$$f_x(x,y)=\frac{\partial}{\partial x}(ye^{x})=y\frac{\partial}{\partial x}(e^{x})=ye^{x}$$

Note: When we differentiate with respect to x, the y just goes outside, as shown above. It's treated like a constant.

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12) $f(x,y)=x^3e^{4y}$

$$f_y(x,y)=\frac{\partial}{\partial y}(x^3e^{4y})=x^3\frac{\partial}{\partial y}(e^{4y})=x^3(e^{4y})(4)=4x^3e^{4y}$$

Notes:

1) Factor x^3 out, and then differentiate with respect to y using the chain rule.

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13) $f(x,y)=x\ln(y)$

$$f_y(x,y)=\frac{\partial}{\partial y}(x\cdot\ln(y))=x\frac{\partial}{\partial y}\ln(y)=x\left(\frac{1}{y}\right)=\frac{x}{y}$$

Notes :
1) Factor x out, and then differntiate with respect to y
using the rule for the natural log function derivative.

14) $f(x,y)=x\ln(xy)$

$$f_x(x,y)=\frac{\partial}{\partial x}(x\cdot\ln(xy))=\left(\frac{\partial}{\partial x}(x)\right)\ln(xy)+x\left(\frac{\partial}{\partial x}\ln(xy)\right)=1\cdot\ln(xy)+x\left(\frac{1}{xy}\right)(y)$$

$$=\ln(xy)+\frac{x}{x}\cdot\frac{y}{y}=\ln(xy)+(1)(1)=\ln(xy)+1$$

Notes:

1) This one relies on the product rule, and the chain rule.

2) Cross off a pair of x's, and a pair of y's. Remember that

$\frac{x}{x}=1$ and $\frac{y}{y}=1$.

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15) $f(x,y)=\cos(xy)$

$$f_x(x,y)=\frac{\partial}{\partial x}\cos(xy)=-\sin(xy)\frac{\partial}{\partial x}(xy)=-\sin(xy)\cdot y=-y\sin(xy)$$

Notes:

1) This one relies on the chain rule. Differentiate cos and then finish by multiplying by the derivative of xy with respect to x.

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16) $f(x,y)=\ln(x^2+y)$

$$f_x(x,y)=\frac{\partial}{\partial x}\ln(x^2+y)=\frac{1}{x^2+y}\cdot\frac{\partial}{\partial x}(x^2+y)=\frac{1}{x^2+y}\cdot\left(\frac{\partial}{\partial x}(x^2)+\frac{\partial}{\partial x}(y)\right)=\frac{1}{x^2+y}\cdot(2x)$$

$$=\frac{2x}{x^2+y}$$

Notes:

1) This one relies on the chain rule. Differentiate ln and then finish by multiplying by the derivative of x^2+y with respect to x.

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17) $f(x,y)=e^{xy}$

$$f_y(x,y)=\frac{\partial}{\partial y}e^{xy}=e^{xy}\frac{\partial}{\partial y}(xy)=e^{xy}x\frac{\partial}{\partial y}(y)=e^{xy}x(1)=xe^{xy}$$

Notes:

1) This one relies on the chain rule. Differentiate e and then finish by multiplying by the derivative of xy with respect to y.

18) $f(x,y)=e^{x^2+y}$

First rewrite the function, using a basic law of exponents.

$$f(x,y)=e^{x^2+y}=e^{x^2}\cdot e^{y}$$

Now differentiate, as shown.

$$f_x(x,y)=\frac{\partial}{\partial x}(e^{x^2}\cdot e^{y})=e^{y}\frac{\partial}{\partial x}(e^{x^2})=e^{y}\left(e^{x^2}\right)(2x)=2xe^{y}e^{x^2}=2xe^{x^2+y}=2xf(x,y)$$

Notes:

1) With respect to x, y stays constant, so factor out e^{y}.

2) Use the chain rule to differentiate e^{x^2}

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19) $f(x,y)=\frac{x}{\sqrt{x^2+y^2}}$

First rewrite the function, as shown below.

$$f(x,y)=\frac{x}{\sqrt{x^2+y^2}}=\frac{x}{\left(x^2+y^2\right)^{\frac{1}{2}}}=x\left(x^2+y^2\right)^{-\frac{1}{2}}$$

Now differentiate, using the product and chain rules.

$$f_x(x,y)=\frac{\partial}{\partial x}(x\left(x^2+y^2\right)^{-\frac{1}{2}})=\left(\frac{\partial}{\partial x}(x)\right)\left(x^2+y^2\right)^{-\frac{1}{2}}+x\left(\frac{\partial}{\partial x}\left(x^2+y^2\right)^{-\frac{1}{2}}\right)$$

$$=1\cdot\left(x^2+y^2\right)^{-\frac{1}{2}}+x\left(\frac{-1}{2}\right)\left(x^2+y^2\right)^{-\frac{1}{2}-1}(2x)=\frac{1}{\sqrt{x^2+y^2}}+\frac{-x}{2}\cdot\left(x^2+y^2\right)^{-\frac{1}{2}-\frac{2}{2}}$$

$$=\frac{1}{\sqrt{x^2+y^2}}+\frac{-x}{2}\cdot\left(x^2+y^2\right)^{-\frac{3}{2}}=\frac{1}{\sqrt{x^2+y^2}}+\frac{-x}{2}\cdot\frac{1}{\sqrt{x^2+y^2}^3}=\frac{1}{\sqrt{x^2+y^2}}-\frac{x}{2\sqrt{x^2+y^2}^3}$$

Notes:

1) You could combine the last two expressions into a single fractions, but that's algebra. Feel free to amuse yourself with that.

20) $f(x,y)=\ln\left(\frac{1}{xy}\right)$

Dif f erentiate us ing the cha in a nd pr oduct rul es .

$$f_x(x,y)=\frac{\partial}{\partial x}\ln\left(\frac{1}{xy}\right)=\frac{1}{\frac{1}{xy}}\frac{\partial}{\partial x}\left(\frac{1}{xy}\right)=\frac{1}{\frac{1}{xy}}\frac{\partial}{\partial x}(xy)^{-1}=\frac{1}{\frac{1}{xy}}(-1)(xy)^{-1-1}(y)=\frac{1}{\frac{1}{xy}}(-1)(xy)^{-2}(y)$$

$$=\frac{1}{1}\cdot\frac{xy}{1}(-1)\cdot\frac{1}{(xy)^2}\cdot y=-(xy)\frac{1}{(xy)(xy)}\cdot y=-1\cdot\frac{xy}{xy}\cdot\frac{1}{(xy)}\cdot y=-1\cdot\frac{xy}{xy}\cdot\frac{1}{x}\cdot\frac{y}{y}=-1\cdot\frac{1}{x}=\frac{-1}{x}$$

Notes:

1) Begin by differentiating ln

2) Then differentiate $\frac{1}{xy}$

3) Then differentiate xy

4) This means you have to apply the chain rule multiple times

5) $\frac{xy}{xy}=1,\quad \frac{y}{y}=1$

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21) $f(x,y)=\ln(\cos(xy))$

Differentiate using the chain rule several times.

$$f_y(x,y)=\frac{\partial}{\partial y}\ln(\cos(xy))=\frac{1}{\cos(xy)}\left(\frac{\partial}{\partial y}\cos(xy)\right)=\frac{1}{\cos(xy)}\cdot(-1)\sin(xy)\cdot\frac{\partial}{\partial y}(xy)$$

$$=\frac{1}{\cos(xy)}\cdot(-1)\sin(xy)\cdot(x)=-1\cdot\frac{\sin(xy)}{\cos(xy)}\cdot x=-1\cdot\tan(xy)\cdot x=-x\tan(xy)$$

Notes:

1) Begin by differentiating ln
2) Then differentiate cos(xy)
3) Then differentiate xy
4) This means you have to apply the chain rule multiple times

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22) $f(x,y)=\tan^{-1}(xy)$

Differentiate using the chain rule, and then then power rule.

$$f_y(x,y)=\frac{\partial}{\partial y}\tan^{-1}(xy)=\frac{1}{1+(xy)^2}\cdot\frac{\partial}{\partial y}(xy)=\frac{1}{1+(xy)^2}\cdot(x)=\frac{x}{1+(xy)^2}=\frac{x}{1+x^2y^2}$$

Notes:

1) Begin by differentiating the inverse tangent
2) Then differentiate xy
3) Remember this basic property of exponents: $(xy)^a=x^ay^a$

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23) $f(x,y)=\dfrac{x}{\cos(y)}$

First rewrite, as shown below.

$f(x,y)=x(\cos(y))^{-1}$

Now differntiate using the power and chain rules.

$$f_y(x,y)=\frac{\partial}{\partial y}x(\cos(y))^{-1}=x\frac{\partial}{\partial y}(\cos(y))^{-1}=x(-1)(\cos(y))^{-2}\frac{\partial}{\partial y}(\cos(y))$$

$$=-x\big(\cos(y)\big)^{-2}(-\sin(y))=-x(\cos(y))^{-2}=\frac{-x(-\sin(y))}{\cos(y)^2}=\frac{x\sin(y)}{\cos(y)^2}$$

Notes:

1) Begin by rewriting
2) Then move the x outside the derivative symbol
3) Then differentiate with respect to y by using the power and chain rules

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24) $f(x,t)=\sin^2(x-\omega t)$

First rewrite, as shown below.

$$f(x,t)=\left(\sin(x-\omega t)\right)^2$$

Now differntiate using the chain rule.

$$f_t(x,t)=\frac{\partial}{\partial t}\left(\sin(x-\omega t)\right)^2=2(\sin(x-\omega t))\frac{\partial}{\partial t}\sin(x-\omega t)$$

$$=2\sin(x-\omega t)\cos(x-\omega t)\frac{\partial}{\partial t}(x-\omega t)=2\sin(x-\omega t)\cos(x-\omega t)(-\omega)$$

$$=-2\omega\sin(x-\omega t)\cos(x-\omega t)$$

Notes:

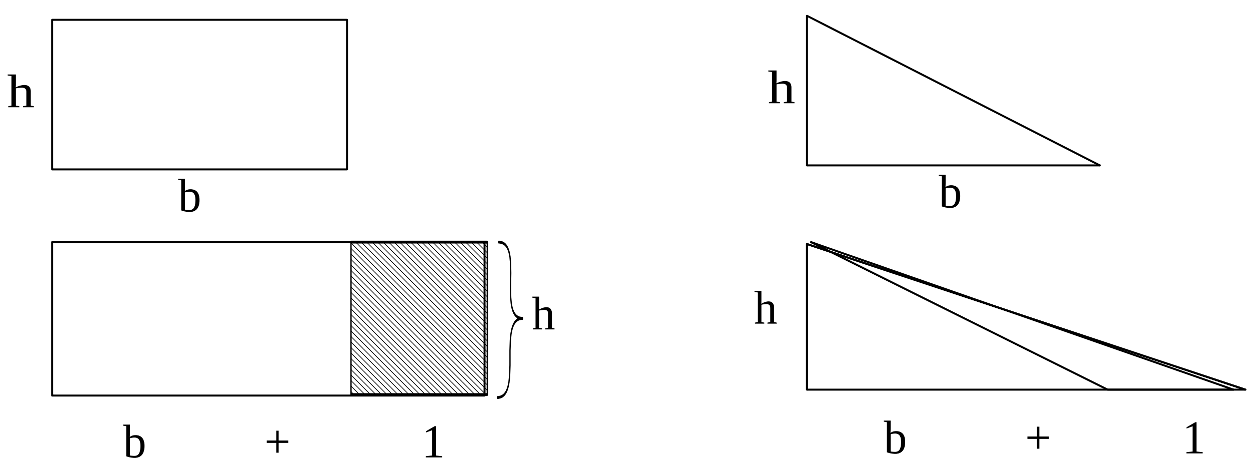
1) Begin by rewriting
2) Then differentiate with respect to t, using the chain rule.

25) $A(b,h)=\frac{1}{2}bh$ Area of a triangle depends on two variables

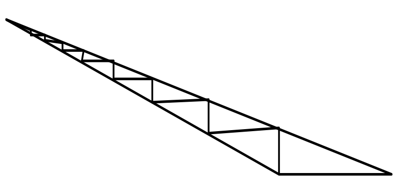
$$A_b(b,h)=\frac{\partial}{\partial b}\left(\frac{1}{2}bh\right)=\frac{1}{2}h\frac{\partial}{\partial b}(b)=\frac{1}{2}h(1)=\frac{1}{2}h$$

Notes:

1) Factor the 1/2 and h out because they're considered constant with respect to b.
2) Then differentiate with respect to b.
3) Look at the picture below. You see that in the case of a rectangle, when b increases by 1, the area increses by h. To be more clear, you see the shaded portion is $h\times 1=h$. Because a triangle is half a rectangle, that rate of change is $\frac{1}{2}h$.



4) In the triangles above, when b is increased by 1, the new area shown appears. I've divided the area into pieces. If we cut the new triangle along the lines shown, and reassemable the pieces, what should we get?



26) $V(r,h)=\pi r^2 h$ Volume of cylinder depends on height and radius

$$V_r(r,h)=\frac{\partial}{\partial r}\left(\pi r^2 h\right)=\pi h\frac{\partial}{\partial r}(r^2)=\pi h(2r)=2\pi hr$$

Notes:

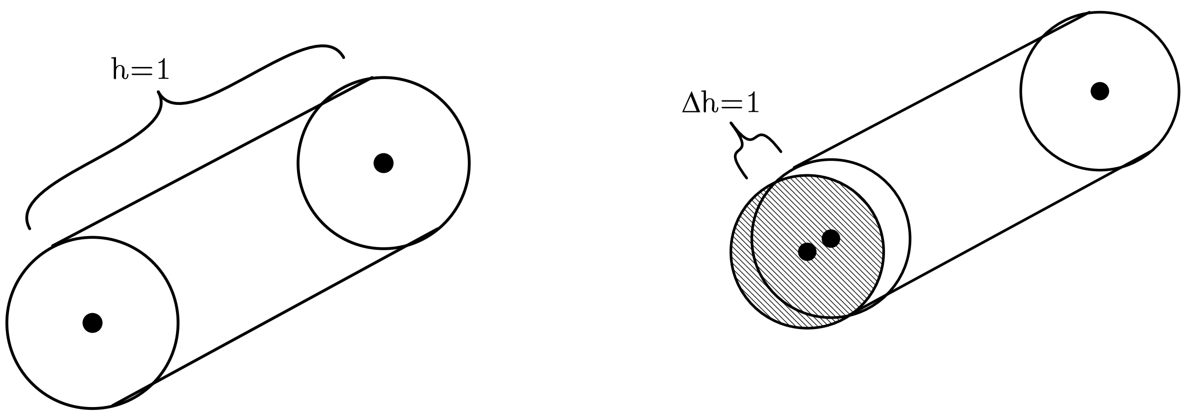
1) Factor the πh because they're considered constant with respect to r.
2) Then differentiate with respect to r.
3) You can imagine this as saying that while we change the radius of the base, the height of the cylinder remains fixed.

27) $V(r,h)=\pi r^2 h$ Volume of cylinder depends on height and radius

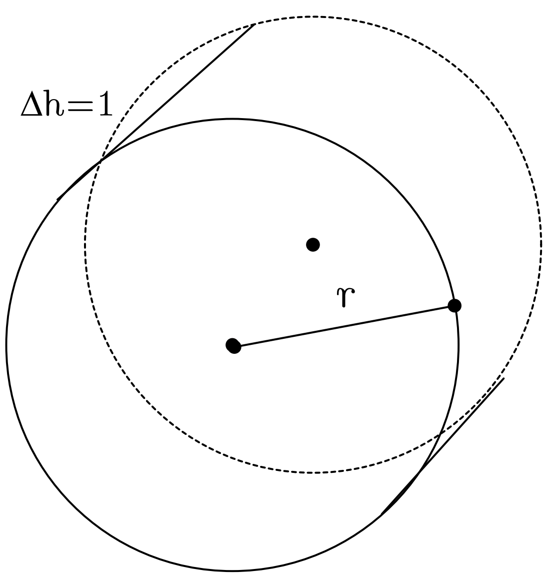
$$V_h(r,h)=\frac{\partial}{\partial h}\left(\pi r^2 h\right)=\pi r^2\frac{\partial}{\partial h}h=\pi r^2(1)=\pi r^2$$

Notes:

1) Do you see what this derivative represents? Think about a famous formula.
2) Here we're changing the height, h, so the base area is fixed. So we factor it from the derivative.



3) As you can see above, when the cylinder grows by 1 in height, the new volume that appears is $\pi r^2(1)=\pi r^2$. This is enlarged below.



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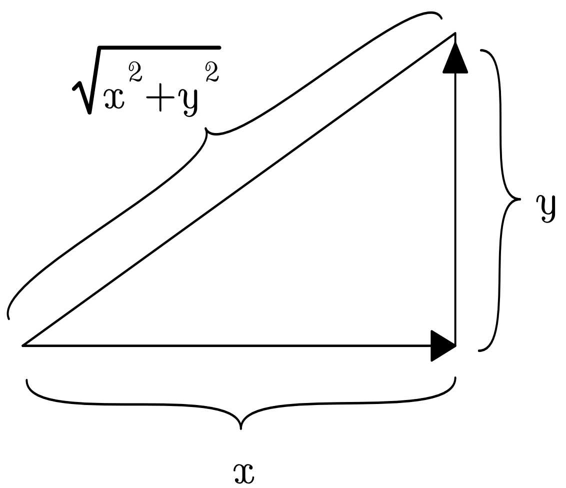
28) $L(x,y)=\sqrt{x^2+y^2}$ Length of a planar vector depends on two variables

$$L_x(x,y)=\frac{\partial}{\partial x}\sqrt{x^2+y^2}=\frac{\partial}{\partial x}(x^2+y^2)^{\frac{1}{2}}=\frac{1}{2}\left(x^2+y^2\right)^{\frac{-1}{2}}(2x)=\frac{2\quad x}{2\sqrt{x^2+y^2}}=\frac{x}{\sqrt{x^2+y^2}}$$



A closer inspection of the expression for the derivative shows that when x changes, the length changes not as x, but as the part that x is of the whole hypotenuse. This makes sense because the length depends both on the horizontal and vertical part.

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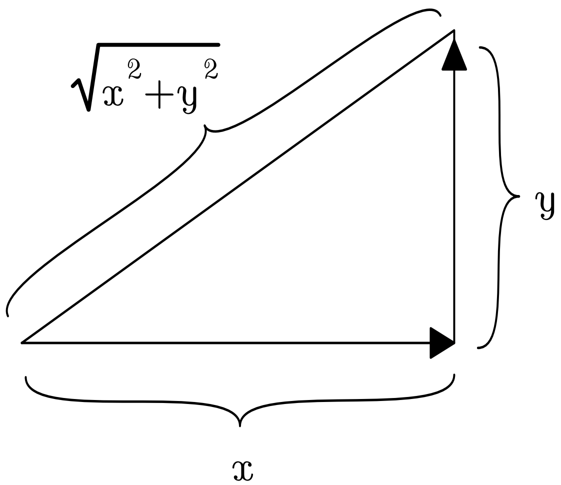
28) $L(x,y)=\sqrt{x^2+y^2}$ Length of a planar vector depends on two variables

$$L_y(x,y)=\frac{\partial}{\partial y}\sqrt{x^2+y^2}=\frac{\partial}{\partial y}(x^2+y^2)^{\frac{1}{2}}=\frac{1}{2}(x^2+y^2)^{\frac{-1}{2}}(2y)=\frac{2y}{2\sqrt{x^2+y^2}}=\frac{y}{\sqrt{x^2+y^2}}$$



A closer inspection of the expression for the derivative shows that when y changes, the length changes not as y, but as the part that y is of the whole hypotenuse. This makes sense because the length depends both on the horizontal and vertical part.

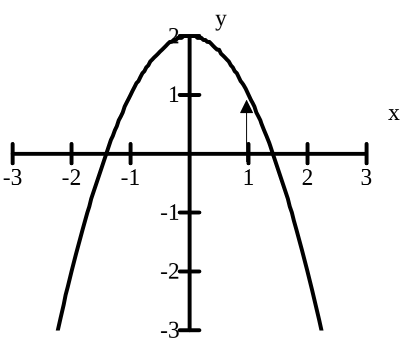
29) $y(x,t)=-(x-2t)^2+2$ A traveling wave

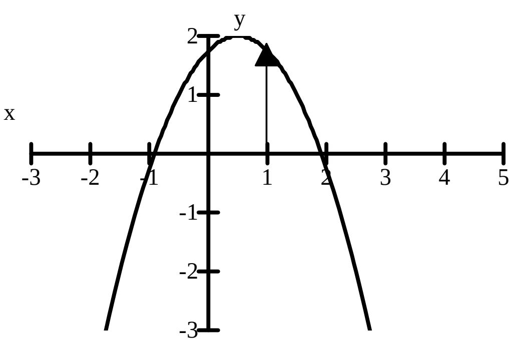
Pretend the position is fixed. This means observe a particle on the wave at various times. Find the rate of change of the height with respect to time. You can imagine this as a vertical line, with the particle moving up and down, depending on the specific wave function.

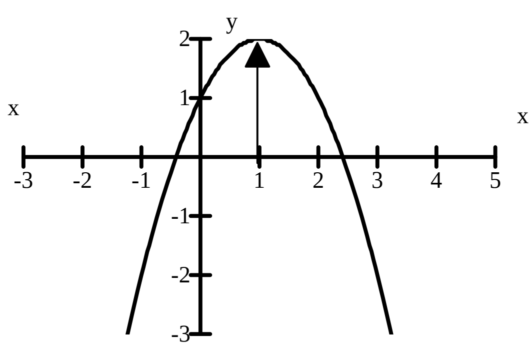
$$\frac{\partial y}{\partial t}y(x,t)=-2(x-2t)(-2)=-2\cdot-2(x-2t)=4(x-2t)=4x-8t$$

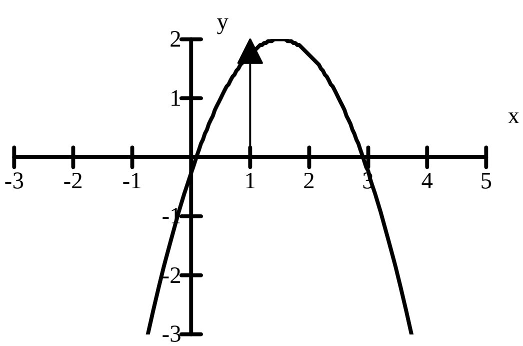
For example, at $x=1$, this quantity is $4(1)-8t=4-8t$.

In the sequence of pictures below, you can see the arrow rise and fall as the wave passes. The position is fixed, so it doesn't shift horizontally.









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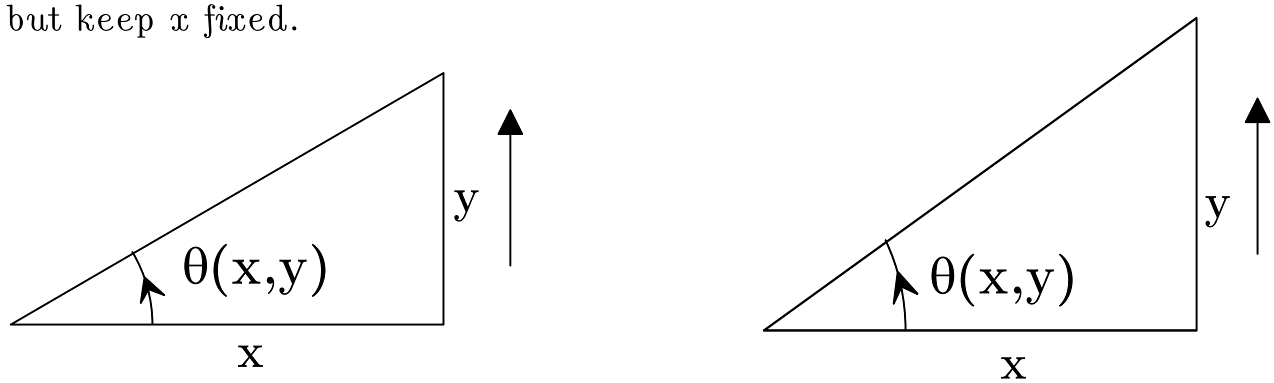
30) $\theta(x,y)=\tan^{-1}\left(\frac{y}{x}\right)$

$$\frac{\partial}{\partial y}\theta(x,y)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{\partial}{\partial y}\left(\frac{y}{x}\right)=\frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)=\left(\frac{1}{1+\frac{y^2}{x^2}}\right)\cdot\left(\frac{1}{x}\right)=\left(\frac{1}{\frac{x^2}{x^2}+\frac{y^2}{x^2}}\right)\cdot\left(\frac{1}{x}\right)=\left(\frac{1}{\frac{x^2+y^2}{x^2}}\right)\cdot\left(\frac{1}{x}\right)$$

$$=\left(\frac{x^2}{x^2+y^2}\right)\cdot\left(\frac{1}{x}\right)=\frac{x}{x^2+y^2}$$

This quantity represents the rate of change of the angle as we vary y, but keep x fixed.



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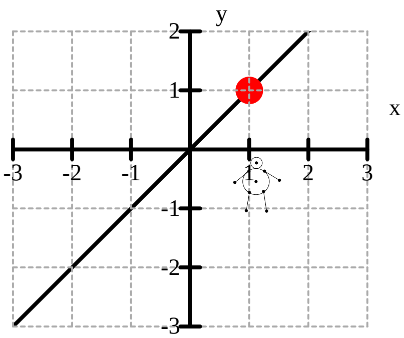
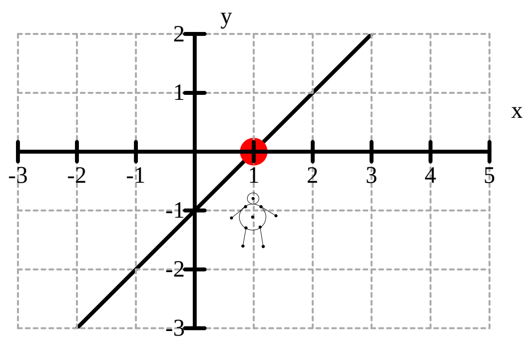
31) $y(x,t)=x-2t$ A traveling distburbance

Pretend the position is fixed. This means observe a particle on the disturbance over a period of time, but keep the position fixed.

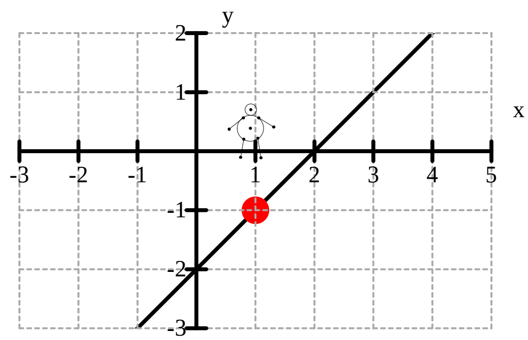
$$\frac{\partial y}{\partial t}y(x,t)=\frac{\partial y}{\partial t}(x-2t)=\frac{\partial y}{\partial t}(x)-\frac{\partial y}{\partial t}(2t)=0-2=-2$$



$y(1,0)=1$



$y(1,\frac{1}{2})=1-2\left(\frac{1}{2}\right)=1-1=0$



$y(1,1)=1-2(1)=1-2=-1$

As you can see in the sequence above, as 1 unit of time passes, the height of the particle decreases by 2 units. Therefore we can conclude there is nothing magical happening here. The little circle figure does not move to the right or left, and this is what we mean by saying the position is fixed at $x=1$.

We can apply the definition of a partial derivative easily here to find the rate at which the height changes as the line moves to the right. Because the x cancels, this tells us that every point on the line moves down 2 units as 1 unit of time passes.

$$\frac{\partial y}{\partial t}=\lim_{h->0}\frac{(x-2(t+h))-(x-2t)}{h}=\lim_{h->0}\frac{(x-2t-2h)-(x-2t)}{h}=\lim_{h->0}\frac{x-2t-2h-(x-2t)}{h}$$

$$=\lim_{h->0}\frac{x-x+2t-2t-2h}{h}=\lim_{h->0}\frac{-2h}{h}=-2$$

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